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Periodic conduction in materials with non-Fourier behaviour

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Abstract

The paper investigates the effects of non-Fourier characteristics of the material on oscillating thermal fields. A general solution in terms of temperature Fourier transform is obtained for an 1-D slab with convective boundary conditions. A transfer matrix of the slab is evaluated and it is shown that its analysis gives information about the non-Fourier behaviour of the material. Periodic conduction is analysed for both harmonic and nonharmonic boundary conditions, and the effect of surface convection is pointed out through the evaluation of its influence on surface temperature phase lag and amplitude. The possibility of evaluating relaxation time from phase lag measurements is pointed out together with the limits imposed by the surface convection. A similar technique is shown to be theoretically applicable to periodic nonharmonic boundary conditions through the evaluation of the temperature-heat flux cross-correlation.

1. Introduction

Fourier law of conduction, combined with the energy equation for rigid bodies, leads to the diffusion equation

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T \tag{1}$$

which has the rather unphysical property of giving rise to instantaneous propagation of thermal pulses. Cattaneo [1,2] was the first to propose a correction to Fourier law, based on the kinetic theory of gases, and later Vernotte [3] proposed independently a similar correction

$$q + \frac{\partial q}{\partial t} t_0 = -k \nabla T \tag{2}$$

where t_0 is the relaxation time. We shall call Eq. (2) by Cattaneo's equation and Fourier law is obtained by setting $t_0 = 0$.

Combining Eq. (2) with the energy equation

$$\rho c \frac{\partial T}{\partial t} = -\nabla q \tag{3}$$

under the hypothesis of constant properties, leads to the hyperbolic heat conduction equation

$$\frac{\partial T}{\partial t} + t_0 \frac{\partial^2 T}{\partial t^2} = \alpha \nabla^2 T \tag{4}$$

which assures a finite propagation velocity of thermal pulses, $\sqrt{\alpha/t_0}$ plays the role of the thermal wave speed. Eq. (4) has the form of the telegraph equation and waves are attenuated by the fact that t_0 is finite (and indeed t_0 is expected to be very small in solids at room temperature, although, to the knowledge of the author, no reliable measurements appear to be available in the literature). Later many other authors derived, in different ways, Eq. (2) (see, for example, Tavernier [4], Muller [5], Meixner [6], Joseph and Preziosi [7] the latter reporting a very interesting and complete bibliographic chronology). Joseph and Preziosi [7] also proposed a modification of Cattaneo law by adding a further term to Eq. (2)

$$q + \frac{\partial q}{\partial t} t_0 = -k \nabla T - k t_1 \nabla \frac{\partial T}{\partial t} \tag{5}$$

which can be written in an alternative form, by using Eq. (3), as

$$q + \frac{\partial q}{\partial t} t_0 - \alpha t_1 \nabla^2 q = -k \nabla T \tag{6}$$

Eq. (4) then becomes

$$\frac{\partial T}{\partial t} + t_0 \frac{\partial^2 T}{\partial t^2} = \alpha \nabla^2 T + \alpha t_1 \nabla^2 \frac{\partial T}{\partial t} \tag{7}$$

t_0 is again a relaxation time whereas t_1 is sometime called "retardation" time. Eq. (5) is often referred as Jeffrey type conduction equation and a deep analysis of this equation and its implications can be found in [7]. These modifications of the Fourier law, while solving the above mentioned paradox

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Nomenclature

Bi	nondimensional convective coefficient (Biot number: $(hL)/k$)
c	specific heat
C	cross-correlation
$E_\pm$	functions
g	function
H	complex coefficient
h	convective heat transfer coefficient
I	complex function
Im{ }	imaginary part
k	thermal conductivity
K	imposed heat flux Fourier transform
L	slab thickness
$\mathbf{M}$	transfer matrix
n	integer number
P	imposed heat flux
q	heat flux
Q	heat flux Fourier transform
$R_\pm$	complex functions
Re{ }	real part
S	temperature Fourier transform
$S_\pm$	complex coefficients
$\mathcal{T}$	time period
T	temperature
t	time
x	position
z	nondimensional variable
$Z_\pm$	complex coefficients for the harmonic case

Greek symbols

α	thermal diffusivity $k/(\rho c)$
β	ratio of retardation and relaxation time t_1/t_0
γ	function
Γ	cross-correlation Fourier transform
δ	Dirac delta-function
Δ	difference
θ	fluctuating temperature
λ	solutions of the associate polynomial
Λ, $\widetilde{\Lambda}$	complex functions
ξ	nondimensional coordinate
Ξ	function
Π	fluctuating part of imposed heat flux
ρ	density
τ	nondimensional time (Fourier number $t\alpha/L^2$)
τ_0	nondimensional relaxation time (Vernotte number $(t_0\alpha)/L^2$)
ϕ	phase lag
Φ	heat flux Fourier transform (harmonic case)
Ψ	complex function
ω	nondimensional frequency, $= (\widetilde{\omega}L^2)/\alpha$
$\widetilde{\omega}$	frequency

Subscripts

a	time average
0	relative to $\xi = 0$
1	relative to $\xi = 1$
g	fluid
r	real part
i	imaginary part

of instantaneous propagation, are not without drawbacks, and criticisms were raised by different authors [8–10].

In the recent years, many papers can be found in the literature dealing with solutions of the hyperbolic heat equation (4), under different boundary conditions (see, for example, [11–20]) or by numerical methods [21–28]. A strong impulse to non-Fourier conduction studies was given by modern application of laser heating, where very rapid heat pulses are generated and possible deviation from Fourier law may produce significative differences in heat propagation (see [29–31], for example). It should be remarked that, apart few well established results for second sound in liquid He and solids at very low temperatures [32–36], conclusive experimental confirmations of deviation from Fourier law in solids at room temperature are still lacking (see, for example, [37]) although some results for inhomogeneous solids are instead available [38,39].

The present paper deals with oscillating thermal fields in a finite 1-D slab, caused by oscillating boundary conditions and, although a general solution for Eq. (7) will be presented below, in all the subsequent analysis the parameter β will be set to zero, thus considering only the Cattaneo case.

The problem of oscillating boundary conditions was already addressed by Tang and Araki [40,41] and B. Abdel-Hamid [42] under imposed harmonic oscillation of surface heat flux. The present paper deals with generally periodic (also nonharmonic) boundary conditions, taking also into account the effect of convection, and suggesting a possible way to detect non-Fourier behaviour by phase measurements (instead of amplitude measurements), an approach implied also by the work of Tang and Araki [40]. The fact that convection is considered allows to show the limits of such possible experimental approach as the convective effect may produce only apparent deviations from Fourier behaviour.

2. A general solution of the 1-D hyperbolic equation

Let now consider the case of a finite 1-D slab having thickness L, in the previous equations then ∇ stands for $\mathrm{d}/\mathrm{d}x$, and introducing the nondimensional coordinate $\xi =$

x/L and time $\tau = t\alpha/L^2$ (τ is often referred as Fourier number) Eq. (7) becomes

$$\frac{\partial T}{\partial \tau} + \tau_0 \frac{\partial^2 T}{\partial \tau^2} = \nabla^2 T + \beta\tau_0 \nabla^2 \frac{\partial T}{\partial \tau} \tag{8}$$

with $\nabla = \mathrm{d}/\mathrm{d}\xi$, $\tau_0 = t_0\alpha/L^2$ (τ_0 is sometime referred as Vernotte number) and $\beta = t_1/t_0$, and Eq. (5) becomes

$$q + \frac{\partial q}{\partial \tau}\tau_0 = -\frac{k}{L}\nabla\left(T + \beta\tau_0 \frac{\partial T}{\partial \tau}\right) \tag{9}$$

The search of oscillating solutions can be pursued by introducing the following transformation:

$$T(\xi,\tau) = T_a(\xi) + \int_{-\infty}^{+\infty} S(\omega,\xi)e^{i\omega\tau}\,\mathrm{d}\omega \tag{10}$$

where $T_a(\xi) = \langle T(\xi,\tau)\rangle = \lim_{\mathcal{T}\to\infty}(1/\mathcal{T})\int_0^{\mathcal{T}} T(\xi,\tau)\,\mathrm{d}\tau$; it is then supposed that the temperature field fluctuates around a well-defined value which may depend only on the position. The nondimensional frequency ω is related to the dimensional frequency $\widetilde{\omega}$ by the relation: $\omega = (\widetilde{\omega}L^2)/\alpha$. By applying the operator $\langle\,\rangle$ to Eq. (8) one obtains:

$$\nabla^2 T_a = 0$$

and substitution of (10) into Eq. (8) gives:

$$\omega\frac{(i - \tau_0\omega)}{(1 + i\beta\omega\tau_0)}S(\omega,\xi) = \frac{\partial^2 S(\omega,\xi)}{\partial \xi^2} \tag{11}$$

The solution of Eq. (11) can be obtained through the associate polynomial equation

$$\lambda^2 = \omega\frac{(i - \tau_0\omega)}{(1 + i\beta\omega\tau_0)}$$

whose roots are

$$\lambda_\pm = \pm\sqrt{\frac{\omega}{2}}g(\omega\tau_0,\beta)\left[\gamma(\omega\tau_0,\beta) + i\gamma^{-1}(\omega\tau_0,\beta)\right]$$

where

$$\gamma(z,\beta) = \sqrt{\frac{\sqrt{1+z^2[1+\beta^2(1+z^2)]}-(1-\beta)z}{1+\beta z^2}}$$

$$g(z,\beta) = \sqrt{\frac{1+\beta z^2}{1+\beta^2 z^2}}$$

and $\gamma(z,\beta)$ is always real for any real z and β; it should also be noticed that $\gamma(z,\beta) = 1$ and $g(z,\beta) = 1$ for the Fourier ($z = 0$, $\beta = 0$) case. Then the general solution of Eq. (11) is

$$S(\omega,\xi) = S_+(\omega)E_+(\omega,\tau_0,\beta,\xi) + S_-(\omega)E_-(\omega,\tau_0,\beta,\xi) \tag{12}$$

where $S_\pm$ are arbitrary complex constants and

$$\begin{aligned}E_\pm(\omega,\tau_0,\beta,\xi) &= e^{\pm\sqrt{\omega/2}g(\omega\tau_0,\beta)[\gamma(\omega\tau_0,\beta)+i\gamma^{-1}(\omega\tau_0,\beta)]\xi} \\ &= e^{\pm\Lambda(\omega,\tau_0,\beta)\xi}\end{aligned}$$

with

$$\Lambda(\omega,\tau_0,\beta) = \sqrt{\omega/2}g(\omega\tau_0,\beta)\left[\gamma(\omega\tau_0,\beta) + i\gamma^{-1}(\omega\tau_0,\beta)\right]$$

The heat flux field can also be split into a steady and a fluctuating component

$$q(\xi,\tau) = q_a(\xi) + \int_{-\infty}^{+\infty} Q(\omega,\xi)e^{i\omega\tau}\,\mathrm{d}\omega \tag{13}$$

with

$$q_a(\xi) = \langle q(\xi,\tau)\rangle = \lim_{\mathcal{T}\to\infty}\frac{1}{\mathcal{T}}\int_0^{\mathcal{T}} q(\xi,\tau)\,\mathrm{d}\tau$$

From Eq. (9)

$$q_a(\xi) = -\frac{k}{L}\frac{\mathrm{d}T_a(\xi)}{\mathrm{d}\xi} \tag{14}$$

and

$$\begin{aligned}Q(\omega,\xi) &= \frac{k}{L}\widetilde{\Lambda}(\omega,\tau_0,\beta) \\ &\quad\times\left[S_-(\omega)E_-(\omega,\tau_0,\beta,\xi) - S_+(\omega)E_+(\omega,\tau_0,\beta,\xi)\right]\end{aligned} \tag{15}$$

with

$$\widetilde{\Lambda}(\omega,\tau_0,\beta) = \Lambda(\omega,\tau_0,\beta)\frac{(1 + i\omega\tau_0\beta)}{(1 + i\omega\tau_0)} \tag{16}$$

A practical way to generate an oscillating thermal field is to impose an oscillating thermal flux on the slab surface. The slab can also exchange heat with the environment (by convection or radiation). Let $P_{0,1}(\tau) = P_{a,0,1} + \Pi_{0,1}(\tau)$ be the heat flux (generated, for example, by a laser beam impinging on the slab surfaces or by a thin foil electrically heated) imposed to the slab surfaces (the subscript is 0 for $\xi = 0$, and 1 for $\xi = 1$) with $\langle P_{0,1}(\tau)\rangle = P_{a,0,1}$ and

$$\Pi_{0,1}(\tau) = \int_{-\infty}^{+\infty} K_{0,1}(\omega)e^{i\omega\tau}\,\mathrm{d}\omega \tag{17}$$

By considering only convective heat transfer on the slab surfaces, the boundary conditions become

$$\begin{aligned}\xi = 0:&\quad P_0(\tau) = q(0,\tau) + h\big(T(0,\tau) - T_g\big) \\ \xi = 1:&\quad P_1(\tau) = -q(1,\tau) + h\big(T(1,\tau) - T_g\big)\end{aligned} \tag{18}$$

Applying the operator $\langle\,\rangle$ to Eq. (18) one obtains:

$$\begin{aligned}\xi = 0:&\quad P_0 = q_a(0) + h\big(T_a(0) - T_g\big) \\ \xi = 1:&\quad P_1 = -q_a(1) + h\big(T_a(1) - T_g\big)\end{aligned} \tag{19}$$

and substituting Eqs. (10) and (15) into (18) and using (19):

$$\begin{aligned}\frac{L}{k}K_0(\omega) &= Q(\omega,0) + Bi\,S(\omega,0) \\ \frac{L}{k}K_1(\omega) &= -Q(\omega,1) + Bi\,S(\omega,1)\end{aligned} \tag{20}$$

then introducing Eqs. (17), (15) and (12) into (20)

$$\begin{aligned}\frac{L}{k}K_0(\omega) &= S_-(\omega,0)\big[Bi + \widetilde{\Lambda}(\omega,\tau_0,\beta)\big] \\ &\quad + S_+(\omega,0)\big[Bi - \widetilde{\Lambda}(\omega,\tau_0,\beta)\big] \\ \frac{L}{k}K_1(\omega) &= S_-(\omega,0)E_-(\omega,\tau_0,\beta,1)\big[Bi - \widetilde{\Lambda}(\omega,\tau_0,\beta)\big] \\ &\quad + S_+(\omega,0)E_+(\omega,\tau_0,\beta,1)\big[Bi + \widetilde{\Lambda}(\omega,\tau_0,\beta)\big]\end{aligned}$$

This system has the following solutions:

$$S_-(\omega) = R_+(\omega, \tau_0, \beta, Bi)E_+(\omega, \tau_0, \beta, 1)K_0(\omega) - R_-(\omega, \tau_0, \beta, Bi)K_1(\omega)$$
$$S_+(\omega) = R_+(\omega, \tau_0, \beta, Bi)K_1(\omega) - R_-(\omega, \tau_0, \beta, Bi)E_-(\omega, \tau_0, \beta, 1)K_0(\omega) \quad (21)$$

where

$$R_+ = \frac{L}{k}\big[Bi + \widetilde{\Lambda}(\omega, \tau_0, \beta)\big] \times \Big(\big[Bi + \widetilde{\Lambda}(\omega, \tau_0, \beta)\big]^2 E_+(\omega, \tau_0, \beta, 1) + \big[Bi - \widetilde{\Lambda}(\omega, \tau_0, \beta)\big]^2 E_-(\omega, \tau_0, \beta, 1)\Big)^{-1}$$
$$R_- = \frac{L}{k}\big[Bi - \widetilde{\Lambda}(\omega, \tau_0, \beta)\big] \times \Big(\big[Bi + \widetilde{\Lambda}(\omega, \tau_0, \beta)\big]^2 E_+(\omega, \tau_0, \beta, 1) + \big[Bi - \widetilde{\Lambda}(\omega, \tau_0, \beta)\big]^2 E_-(\omega, \tau_0, \beta, 1)\Big)^{-1} \quad (22)$$

finally, solution (12) can be written as

$$S(\omega, \xi) = K_0(\omega)I_0(\omega, \tau_0, \beta, Bi, \xi) + K_1(\omega)I_1(\omega, \tau_0, \beta, Bi, \xi) \quad (23)$$

with

$$I_0(\omega, \tau_0, \beta, Bi, \xi) = R_+(\omega, \tau_0, \beta, Bi)E_+(\omega, \tau_0, \beta, 1-\xi) - R_-(\omega, \tau_0, \beta, Bi)E_-(\omega, \tau_0, \beta, 1-\xi)$$
$$I_1(\omega, \tau_0, \beta, Bi, \xi) = R_+(\omega, \tau_0, \beta, Bi)E_-(\omega, \tau_0, \beta, \xi) - R_-(\omega, \tau_0, \beta, Bi)E_+(\omega, \tau_0, \beta, \xi) \quad (24)$$

It is then interesting to observe that Eq. (23) can be written in a vectorial form as

$$S(\omega, \xi) = \big[\, I_0(\omega, \xi) \;\; I_1(\omega, \xi) \,\big] \begin{bmatrix} K_0(\omega) \\ K_1(\omega) \end{bmatrix} \quad (25)$$

where the vector $\mathbf{M}(\omega, \xi) = [\, I_0(\omega, \xi) \;\; I_1(\omega, \xi) \,]$ can be considered as a transfer matrix of the "system slab", and it depends also on τ_0, β, Bi, that means that non-Fourier material behaviour influences the slab response to oscillating inputs. As above pointed out, in the following only the case $\beta = 0$ will be analysed and all the functional dependences on β will be dropped.

3. Harmonic thermal fields

Harmonic solutions can be found by choosing

$$S_\pm(\omega) = Z_\pm(\omega)\delta(\overline{\omega} - \omega)$$

where $\delta(x)$ is the Dirac δ-function (see [43]), then

$$\theta(\xi, \tau) = T(\xi, \tau) - T_a(\xi) = \mathrm{Re}\left\{ \int_{-\infty}^{+\infty} Z(\omega, \xi)\delta(\overline{\omega} - \omega)e^{i\omega\tau}\, \mathrm{d}\omega \right\} = Z_r(\overline{\omega}, \xi)\cos(\overline{\omega}\tau) - Z_i(\overline{\omega}, \xi)\sin(\overline{\omega}\tau) \quad (26)$$

with

$$Z(\overline{\omega}, \xi) = Z_+(\overline{\omega})E_+(\overline{\omega}, \tau_0, \xi) + Z_-(\overline{\omega})E_-(\omega, \tau_0, \xi) \quad (27)$$

and

$$Z_r(\overline{\omega}, \xi) = \mathrm{Re}\big\{Z(\overline{\omega}, \xi)\big\}$$
$$Z_i(\overline{\omega}, \xi) = \mathrm{Im}\big\{Z(\overline{\omega}, \xi)\big\}$$

Again, for the heat flux $Q(\omega, \xi) = \Phi(\omega, \xi)\delta(\overline{\omega} - \omega)$, then

$$\varphi(\xi, \tau) = q(\xi, \tau) - q_a(\xi) = \mathrm{Re}\left\{ \int_{-\infty}^{+\infty} \Phi(\omega, \xi)\delta(\overline{\omega} - \omega)e^{i\omega\tau}\, \mathrm{d}\omega \right\} = \Phi_r(\overline{\omega}, \xi)\cos(\overline{\omega}\tau) - \Phi_i(\overline{\omega}, \xi)\sin(\overline{\omega}\tau) \quad (28)$$

with

$$\Phi(\overline{\omega}, \xi) = \frac{k}{L}\widetilde{\Lambda}(\overline{\omega}, \tau_0, \beta) \times \big[Z_-(\overline{\omega})E_-(\overline{\omega}, \tau_0, \xi) - Z_+(\overline{\omega})E_+(\overline{\omega}, \tau_0, \xi)\big] \quad (29)$$

and

$$\Phi_r(\overline{\omega}, \xi) = \mathrm{Re}\big\{\Phi(\overline{\omega}, \xi)\big\}$$
$$\Phi_i(\overline{\omega}, \xi) = \mathrm{Im}\big\{\Phi(\overline{\omega}, \xi)\big\}$$

The phase lag between the oscillating temperature field at different locations $\phi(\xi, \overline{\omega})$ and the oscillation amplitude $A(\overline{\omega}, \xi)$ can now be evaluated by setting $\theta(\xi, \overline{\omega}) = A(\overline{\omega}, \xi)\sin[\overline{\omega}\tau + \phi(\xi, \overline{\omega})]$, where

$$A(\overline{\omega}, \xi) = \sqrt{Z_r^2(\overline{\omega}, \xi) + Z_i^2(\overline{\omega}, \xi)}$$
$$\tan\big[\phi(\xi, \overline{\omega})\big] = -\frac{Z_r(\overline{\omega}, \xi)}{Z_i(\overline{\omega}, \xi)} \quad (30)$$

It should be remarked that both $A(\overline{\omega}, \xi)$ and $\phi(\xi, \overline{\omega})$ depend also on τ_0.

From Eq. (12) it is easy to get

$$Z_r(\overline{\omega}, \xi) = \big[Z_+^{(r)}(\overline{\omega})e^{\sqrt{\overline{\omega}/2}g\gamma\xi} + Z_-^{(r)}(\overline{\omega})e^{-\sqrt{\overline{\omega}/2}g\gamma\xi}\big] \times \cos\left(\sqrt{\frac{\overline{\omega}}{2}}g\gamma^{-1}\xi\right) + \big[Z_-^{(i)}(\omega)e^{-\sqrt{\overline{\omega}/2}g\gamma\xi} - Z_+^{(i)}(\omega)e^{\sqrt{\overline{\omega}/2}g\gamma\xi}\big] \times \sin\left(\sqrt{\frac{\overline{\omega}}{2}}g\gamma^{-1}\xi\right)$$
$$Z_i(\overline{\omega}, \xi) = \big[Z_+^{(r)}(\overline{\omega})e^{\sqrt{\overline{\omega}/2}g\gamma\xi} - Z_-^{(r)}(\overline{\omega})e^{-\sqrt{\overline{\omega}/2}g\gamma\xi}\big] \times \sin\left(\sqrt{\frac{\overline{\omega}}{2}}g\gamma^{-1}\xi\right)$$

$$+\left[Z_+^{(i)}(\varpi)e^{\sqrt{\varpi/2}g\gamma\xi}+Z_-^{(i)}(\varpi)e^{-\sqrt{\varpi/2}g\gamma\xi}\right]\times\cos\left(\sqrt{\frac{\varpi}{2}}g\gamma^{-1}\xi\right)$$

the four functions $Z_\pm^{(r)}(\varpi)$, $Z_\pm^{(i)}(\varpi)$ can be evaluated from the boundary conditions. Let again $P_{0,1}(\tau)=P_{a,0,1}+\Pi_{0,1}(\tau)$ be the heat flux imposed to the slab surfaces (the subscript is 0 for $\xi=0$, and 1 for $\xi=1$) with $\langle P_{0,1}(\tau)\rangle=P_{a,0,1}$ and for the harmonic case let set

$$\begin{aligned}\Pi_{0,1}(\tau)&=\mathrm{Re}\left\{\int_{-\infty}^{+\infty}K_{0,1}(\omega)\delta(\varpi-\omega)e^{i\omega\tau}\,\mathrm{d}\omega\right\}\\&=K_{0,1}^{(r)}(\varpi)\cos(\varpi\tau)-K_{0,1}^{(i)}(\varpi)\sin(\varpi\tau)\\&=H_{0,1}(\varpi)\sin(\varpi\tau+\phi_{0,1})\end{aligned}\tag{31}$$

where

$$K_{0,1}^{(i)}(\varpi)=\mathrm{Im}\{K_{0,1}(\varpi)\}=-H_{0,1}(\varpi)\cos(\phi_{0,1})$$
$$K_{0,1}^{(r)}(\varpi)=\mathrm{Re}\{K_{0,1}(\varpi)\}=H_{0,1}(\varpi)\sin(\phi_{0,1})$$

Without loss of generality, one can put $\phi_0=0$ and $\phi_1=\phi$, thus

$$\begin{aligned}&K_0^{(r)}(\varpi)=0\\&K_0^{(i)}(\varpi)=-H_0(\varpi)\\&K_1^{(r)}(\varpi)=H_1(\varpi)\sin(\phi)\\&K_1^{(i)}(\varpi)=-H_1(\varpi)\cos(\phi)\end{aligned}\tag{32}$$

By applying the boundary conditions, Eq. (21) becomes

$$\begin{aligned}Z_-(\varpi)&=R_+(\varpi,\tau_0,Bi)E_+(\varpi,\tau_0,1)K_0(\varpi)\\&\quad-R_-(\varpi,\tau_0,Bi)K_1(\varpi)\\Z_+(\varpi)&=R_+(\varpi,\tau_0,Bi)K_1(\varpi)\\&\quad-R_-(\varpi,\tau_0,Bi)E_-(\varpi,\tau_0,1)K_0(\varpi)\end{aligned}$$

and the temperature field on the surfaces can be evaluated from the equations

$$Z(\varpi,0)=K_0(\varpi)I_0(\varpi,\tau_0,Bi,0)+K_1(\varpi)I_1(\varpi,\tau_0,Bi,0)$$
$$Z(\varpi,1)=K_0(\varpi)I_0(\varpi,\tau_0,Bi,1)+K_1(\varpi)I_1(\varpi,\tau_0,Bi,1)$$

4. Non-Fourier effects for the harmonic case

To observe some features of the harmonic case it is interesting to evaluate the phase lags

$$\phi(0,\varpi)=\arctan\left[-\frac{Z_r(\varpi,0)}{Z_i(\varpi,0)}\right]$$
$$\phi(1,\varpi)=\arctan\left[-\frac{Z_r(\varpi,1)}{Z_i(\varpi,1)}\right]$$

for a simpler situation. Let $K_1(\varpi)=0$, that means that the imposed heat flux on the surface $\xi=1$ is steady (but $P_{a,1}$ can be different from zero). As in this case $K_0(\varpi)=-iH_0(\varpi)$ (where $H_0(\varpi)$ is real)

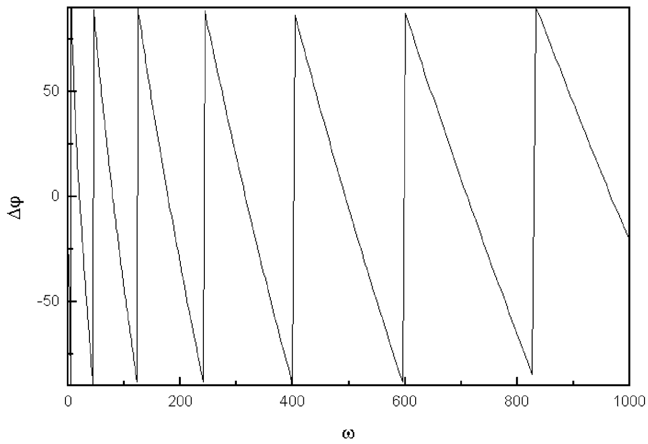

Fig. 1. Surface temperature phase lag for the Fourier case ($\tau_0=0$) without convection $Bi=0$.

$$Z(\varpi,0)=-H_0(\varpi)iI_0(\varpi,\tau_0,Bi,0)$$
$$Z(\varpi,1)=-H_0(\varpi)iI_0(\varpi,\tau_0,Bi,1)$$

and

$$\begin{aligned}Z_r(\varpi,\xi)&=\mathrm{Re}\{-H_0(\varpi)iI_0(\varpi,\tau_0,Bi,\xi)\}\\&=H_0(\varpi)I_{0,i}(\varpi,\tau_0,Bi,\xi)\\Z_i(\varpi,\xi)&=\mathrm{Im}\{-H_0(\varpi)iI_0(\varpi,\tau_0,Bi,\xi)\}\\&=-H_0(\varpi)I_{0,r}(\varpi,\tau_0,Bi,\xi)\end{aligned}$$

the phase lag assumes the form

$$\begin{aligned}\phi(\xi,\varpi,\tau_0,Bi)&=\arctan\left[-\frac{Z_r(\varpi,\xi)}{Z_i(\varpi,\xi)}\right]\\&=\arctan\left[\frac{I_{0,i}(\varpi,\tau_0,Bi,\xi)}{I_{0,r}(\varpi,\tau_0,Bi,\xi)}\right]\end{aligned}$$

clearly independent of the amplitude of the imposed oscillating flux. The phase lag between the surface temperatures

$$\Delta\phi_{1,0}(\varpi,\tau_0,Bi)=\phi(1,\varpi,\tau_0,Bi)-\phi(0,\varpi,\tau_0,Bi)$$

is then, for a given frequency ϖ, a function of τ_0 and *Bi*, and non-Fourier effects may be observed through the evaluation of such lag. Fig. 1 shows $\Delta\phi$ for the Fourier case ($\tau_0=0$) and with $Bi=0$. As expected it depends on the frequency of the harmonic oscillation. Obviously the presence of a convective heat transfer on the surfaces influences the phase lag and Fig. 2 reports the influence of *Bi* on the phase lag showing the difference $\varphi_{\mathrm{F}}=\Delta\phi_{1,0}(\varpi,0,Bi)-\Delta\phi_{1,0}(\varpi,0,0)$ (i.e., for the Fourier case). The effect begins to become observable for $Bi>0.1$. Fig. 3 shows instead the non-Fourier effects for different values of τ_0 (but with $Bi=0$), the plotted variable is $\varphi_C(\varpi,\tau_0)=\Delta\phi_{1,0}(\varpi,\tau_0,0)-\Delta\phi_{1,0}(\varpi,0,0)$, i.e., the phase lag variation imposed by non-Fourier behaviour. It is interesting to observe that in this case the variable φ_C appears to depend on $\varpi\cdot\tau_0^{2/3}$,

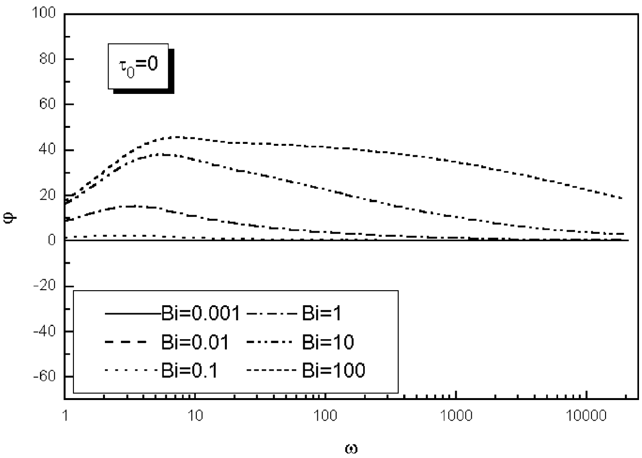

Fig. 2. Influence of convection on the surface temperature phase lag for the Fourier case.

particularly for small values of τ_0, as it is shown in Fig. 3(b). This suggests that the effect may become measurable (for example, by a variation of the phase lag of a few degree) for nondimensional frequencies $\overline{\omega}$ proportional to $1/\tau_0^{2/3}$, for example, $\varphi_C = \Delta\phi_{1,0}(\overline{\omega}, \tau_0, 0) - \Delta\phi_{1,0}(\overline{\omega}, 0, 0) = 5°$ for $\overline{\omega} = 0.4/\tau_0^{2/3}$. Finally, Fig. 4(a) and (b) show the effect of $Bi \neq 0$ on the non-Fourier cases with $\tau_0 = 10^{-6}$ and $\tau_0 = 10^{-4}$. Again, the effect of convection becomes evident only for Bi larger than 0.1. It is, however, evident that to detect non-Fourier behaviour of a material by evaluating the phase lag, Bi must be kept small, condition achievable by both keeping a small value of the convective coefficient (h) and a small slab thickness (L).

5. The periodic nonharmonic case

Harmonic periodic heating is a condition not easy to achieve in practice; it is much easier to obtain periodic

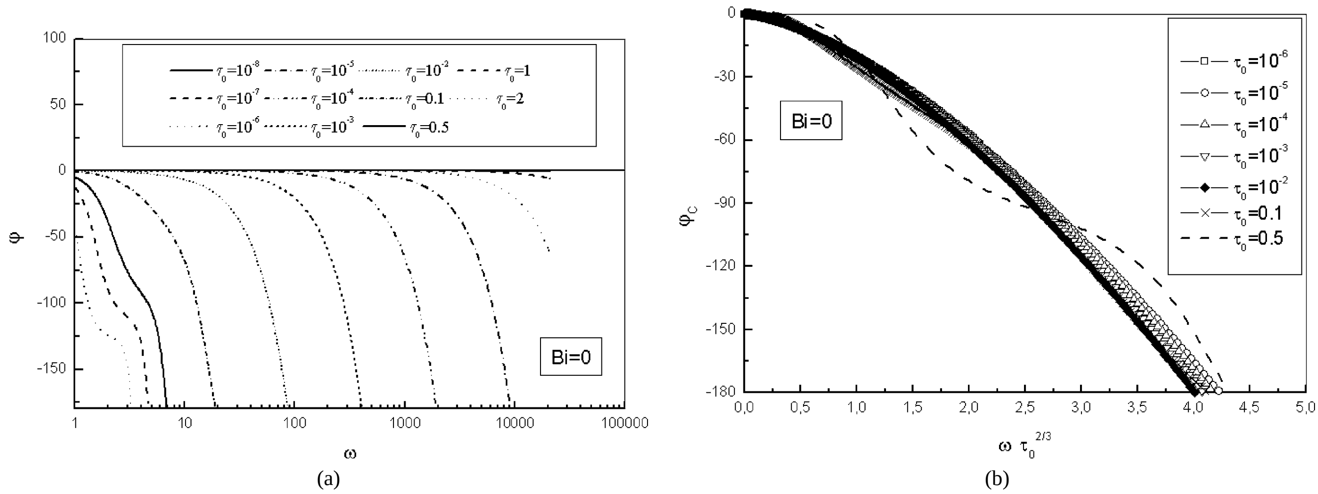

Fig. 3. (a) Non-Fourier effects on surface temperature phase lag for different Vernotte number without convection ($Bi = 0$). (b) Functional dependence of surface temperature phase lag for the non-Fourier case without convection ($Bi = 0$).

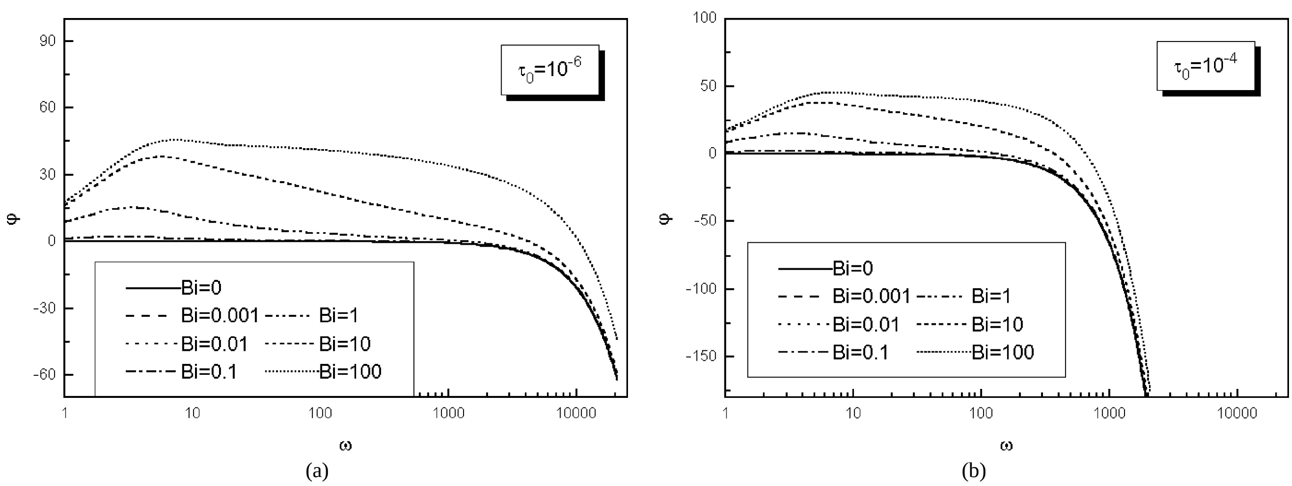

Fig. 4. Effect of convection on surface temperature phase lag in non-Fourier conduction for two values of the nondimensional relaxation time: (a) $\tau_0 = 10^{-6}$; (b) $\tau_0 = 10^{-4}$.

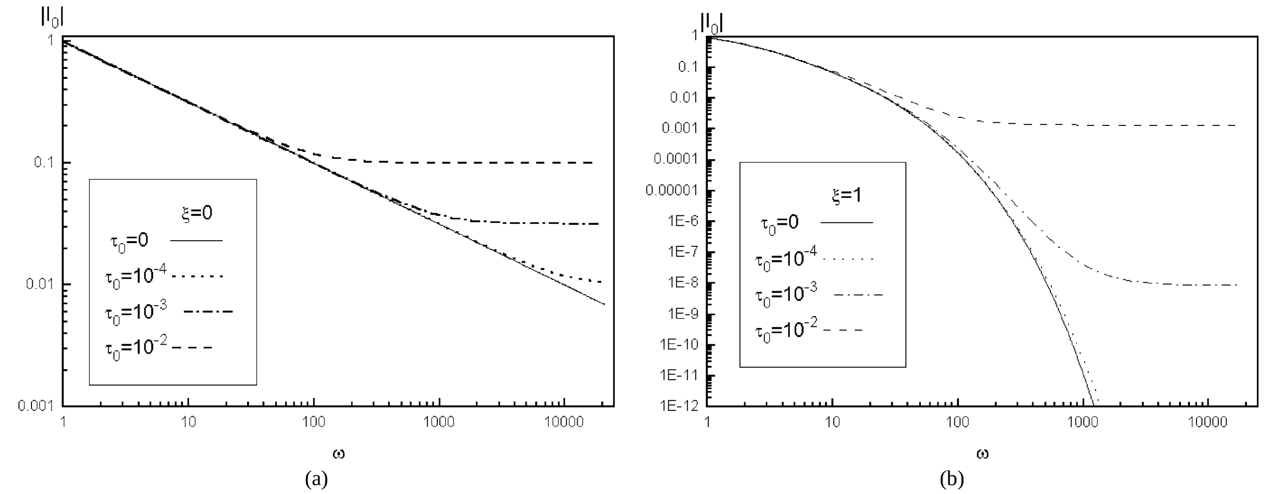

Fig. 5. Transfer function I_0 for different relaxation times (Eq. (24) with $\beta = 0$, and $Bi = 0$).

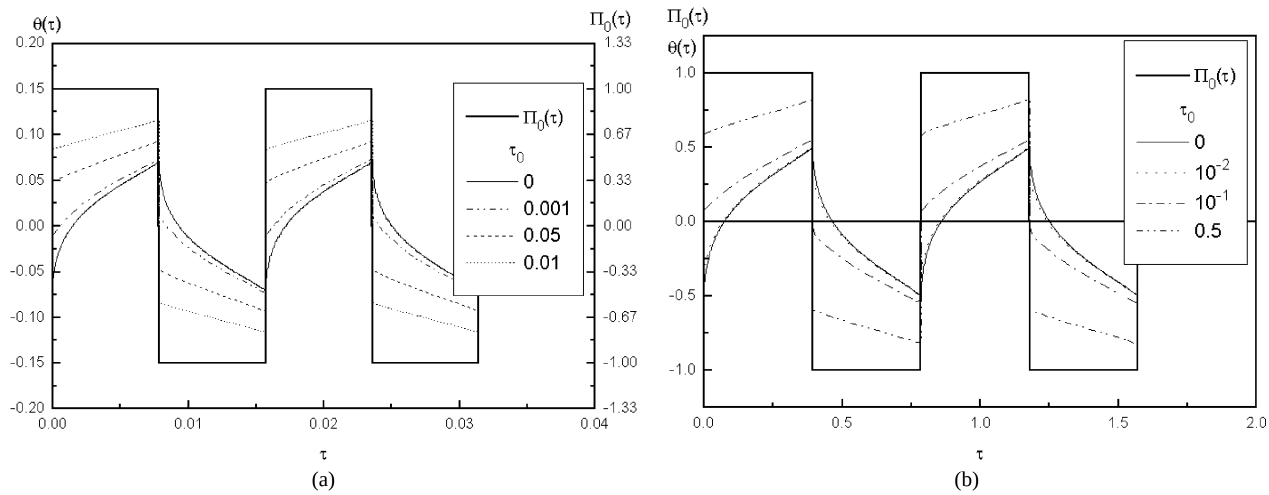

Fig. 6. Temperature fluctuation for a given input function ($\Pi_0(t)$) for different Vernotte numbers and for two different periods of the input function.

nonharmonic heating (for example, by pulsed laser heating, etc.). Let again consider the simpler situation achievable by heating only one surface, so that again $K_1(\overline{\omega}) = 0$. The solution (25) can then be written as

$$S(\omega, \xi) = \left[I_0(\omega, \xi)\ I_1(\omega, \xi) \right] \begin{bmatrix} K_0(\omega) \\ K_1(\omega) \end{bmatrix} = I_0(\omega, \xi) K_0(\omega) \tag{33}$$

The transfer function $I_0(\omega, \xi)$ acts on the "input" signal both by damping the oscillation amplitude and changing the wave shape. The transfer functions magnitudes $|I_0(\omega, 0)|$ and $|I_0(\omega, 1)|$ (relating the input fluctuation to the surfaces temperature fluctuations) are reported in Fig. 5 for $Bi = 0$. As expected, the smaller τ_0, the larger the frequency at which the non-Fourier effects become observable. The magnitudes flatten out after a certain frequency, showing that the larger frequency components of the oscillating fields are not damped in the Cattaneo case, whereas they become vanishingly small for the Fourier case. It is of a certain interest to notice that the asymptotic behaviour of I_0 (and I_1) for $\omega \to \infty$ is oscillating (see Appendix A) for all $\xi \neq 0$ ($\xi \neq 1$) around the value $-\sqrt{\tau_0}$. To better appreciate these facts, let us consider an input heating of the form

$$\Pi_0(\tau) = \begin{cases} B_0, & \text{for } 2n\mathcal{T} \leqslant \tau < (2n+1)\mathcal{T} \\ 0, & \text{for } (2n+1)\mathcal{T} \leqslant \tau < (2n+2)\mathcal{T} \end{cases}$$

for $n = 0, \pm 1, \ldots$, where $2\mathcal{T}$ is the period (see Fig. 6(a)). The input $K_0(\omega)$ can be calculated from Eq. (17) and the temperature fluctuation can be calculated as well (from Eq. (10)). The shape of the temperature fluctuations (in $\xi = 0$) reported in Fig. 6(a) and (b) (for two different values of the input signal period $\mathcal{T}$) show the usual damped response for the Fourier case, as τ_0 increases a sharper response is observed as the damping of the larger frequency become less effective (see also Fig. 5).

It is also interesting to observe that the cross-correlation between the input signal and the surface temperature at $\xi = 0$

$$C(\tau) = \frac{1}{T} \lim_{T \to \infty} \int_0^T \theta(t, 0) \Pi_0(t + \tau)\, \mathrm{d}t$$

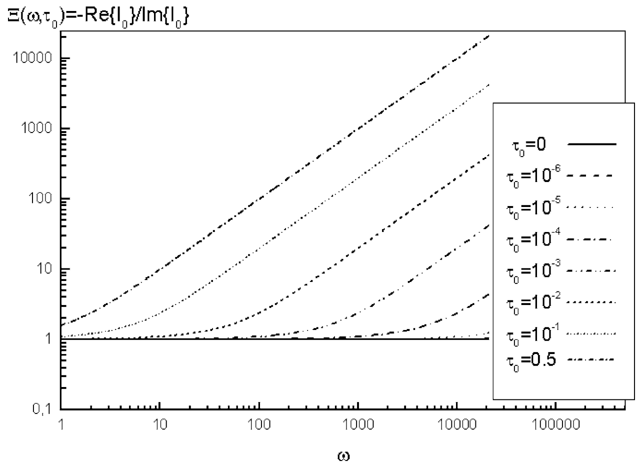

Fig. 7. The function $\Xi(\omega, \tau_0)$ for different Vernotte numbers.

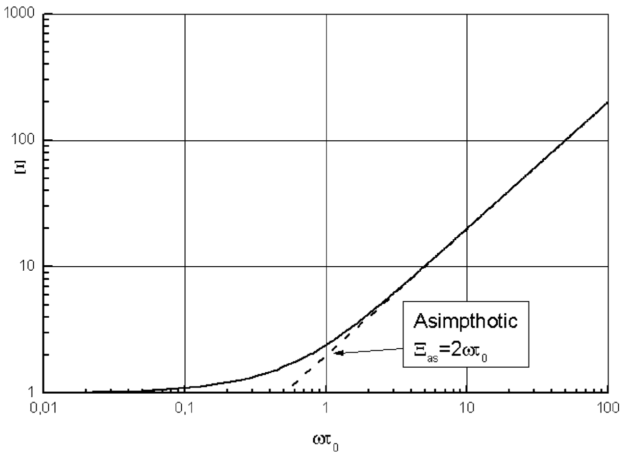

Fig. 8. Functional dependence of $\Xi(\cdot)$ and its asymptotic behaviour.

has a Fourier transform $\Gamma_0(\omega)$

$$C(\tau) = \int_{-\infty}^{+\infty} \Gamma_0(\omega) e^{i\omega\tau} \, d\omega$$

related to $K_0(\omega)$ and $S(0, \omega)$ by the relation (see, for example, [43,44])

$$\Gamma_0(\omega) = S(\omega, 0) K_0^*(\omega)$$

and then

$$\Gamma_0(\omega) = S(\omega, 0) K_0^*(\omega) = I_0(\omega, 0) \left| K_0(\omega) \right|^2$$

so that

$$\Xi(\omega, \tau_0) = -\frac{\mathrm{Re}\{\Gamma_0(\omega)\}}{\mathrm{Im}\{\Gamma_0(\omega)\}} = -\frac{\mathrm{Re}\{I_0(\omega, 0)\}}{\mathrm{Im}\{I_0(\omega, 0)\}}$$

is independent of the actual shape of the input signal $\Pi_0(t)$ and of the surface temperature response $\theta(t, 0)$, but only depends on the material characteristics, hidden into $I_0(\omega, 0)$. Fig. 7 shows the function $\Xi(\omega)$ for different values of the Vernotte number τ_0 for the case $Bi = 0$, but it easy to show (see Appendix C) that $\Xi(\omega, \tau_0)$ can be expressed (for $Bi = 0$) as a function of $\omega\tau_0$ and $\Xi(\omega\tau_0) = \gamma^{-2}(\omega\tau_0)$ (see Fig. 8). This result shows how it is possible to observe non-Fourier behaviour of a material by evaluating the cross-correlation between the imposed heat flux and the surface temperature, independently of the actual shape of the heating pulses.

6. Conclusions

The paper investigates the effects of non-Fourier characteristics of the material on oscillating thermal fields in a 1-D slab and the following conclusions were obtained:

– A general solution in terms of Fourier transform was obtained for an 1-D slab with convective boundary conditions and the "transfer function" of the slab was evaluated, showing that the study of this characteristic function can give information about the non-Fourier behaviour of the material.
– Harmonic oscillating thermal fields were analysed and some peculiar characteristics were evidenced, like the dependence of the phase lag between surface temperatures on Vernotte number and convective boundary conditions. This would allow to evaluate the relaxation time from phase lag measurements, usually more feasible than intensity (direct temperature) measurements, and the present findings show the limit imposed to this possible technique by the effects of convection.
– Nonharmonic periodic thermal fields, obtainable by periodic heating of the material surface, were analysed and the effect of non-Fourier behaviour on the oscillating temperature profiles was pointed out. It was also shown that the Fourier components of the cross-correlation between the input heat-flux and the surface temperature can be used to evaluate the relaxation time of a material, independently of the actual shape of the heating pulses.

Appendix A

The phase lag $\phi(0, \varpi)$ for the Fourier case ($\tau_0 = 0, \beta = 0$) and for, $Bi = 0$ is

$$\phi(0, \varpi, 0, 0, 0) = \arctan\left[\frac{I_0^{(i)}(\varpi, 0, 0, 0, 0)}{I_0^{(r)}(\varpi, 0, 0, 0, 0)}\right]$$

but

$$I_0(\omega, 0, 0, 0, 0) = \big[R_+(\omega, 0, 0, 0) E_+(\omega, 0, 1) - R_-(\omega, 0, 0, 0) E_-(\omega, 0, 1)\big]$$

and

$$R_+ = \frac{L}{k} \frac{1}{\widetilde{\Lambda}(\omega, 0, 0)[E_+(\omega, 0, 1) + E_-(\omega, 0, 1)]}$$

$$R_- = \frac{L}{k} \frac{-1}{\widetilde{\Lambda}(\omega, 0, 0)[E_+(\omega, 0, 1) + E_-(\omega, 0, 1)]} = -R_+$$

so that

$$I_0(\omega,0)=R_+(\omega)\big[E_+(\omega,0,1)+E_-(\omega,0,1)\big]$$
$$=\frac{L}{k}\frac{[E_+(\omega,0,1)+E_-(\omega,0,1)]}{\widetilde{\Lambda}(\omega,0,0)[E_+(\omega,0,1)+E_-(\omega,0,1)]}$$
$$=\frac{L}{k\widetilde{\Lambda}(\omega,0,0)}$$

For the Fourier case

$$\Lambda(\omega,\tau_0,\beta)=\sqrt{\frac{\omega}{2}}[1+i]$$

so that

$$I_0(\omega,0)=\frac{L}{k}\frac{1}{\widetilde{\Lambda}(\omega,0,0)}=\frac{L}{k}\frac{1-i}{\sqrt{\omega/2}\,2}$$

that means that the phase lag $\phi(0,\overline{\omega})$ is always equal to

$$\phi(0,\overline{\omega})=\arctan[-1]=-\frac{\pi}{4}$$

Appendix B

The asymptotic behaviour for $\omega\to\infty$ of the functions I_0 and I_1 can be evaluated for the case with $\beta=0$ and with $Bi=0$ (the last condition can be understood as a limiting case for $Bi\ll\sqrt{\omega/2}$). Let find first the asymptotic behaviour of γ

$$\lim_{\omega\to\infty}\gamma(\omega\tau_0)=\lim_{z\to\infty}\sqrt{\sqrt{1+z^2}-z}$$
$$=\lim_{z\to\infty}\sqrt{z}\sqrt{\sqrt{1+\frac{1}{z^2}}-1}$$
$$=\lim_{z\to\infty}\sqrt{z}\sqrt{1+\frac{1}{2z^2}-1}=\frac{1}{\sqrt{2z}} \tag{B.1}$$

with $z=\omega\tau_0$. Then from Eq. (22) (with $Bi=0$)

$$R_+(\omega,\tau_0)=\frac{L}{k}\frac{1}{\widetilde{\Lambda}(\omega,\tau_0,\beta)[E_+(\omega,\tau_0,1)+E_-(\omega,\tau_0,1)]}$$
$$R_-(\omega,\tau_0)=\frac{L}{k}\frac{-1}{\widetilde{\Lambda}(\omega,\tau_0,\beta)[E_+(\omega,\tau_0,1)+E_-(\omega,\tau_0,1)]} \tag{B.2}$$

and from Eq. (24)

$$I_0(\omega,\tau_0,\xi)=R_+(\omega,\tau_0)\big[E_+(\omega,\tau_0,1-\xi)+E_-(\omega,\tau_0,1-\xi)\big]$$
$$I_1(\omega,\tau_0,\xi)=R_+(\omega,\tau_0)\big[E_-(\omega,\tau_0,\beta,\xi)+E_+(\omega,\tau_0,\beta,\xi)\big]$$

or, using Eqs. (16) and (B.2) and the definition of Λ

$$I_0(\omega,\tau_0,\xi)=\frac{L}{k}\frac{(1+i\omega\tau_0)[E_+(\omega,\tau_0,1-\xi)+E_-(\omega,\tau_0,1-\xi)]}{\sqrt{\omega/2}[\gamma+i\gamma^{-1}][E_+(\omega,\tau_0,1)+E_-(\omega,\tau_0,1)]}$$
$$I_1(\omega,\tau_0,\xi)=\frac{L}{k}\frac{(1+i\omega\tau_0)[E_+(\omega,\tau_0,\xi)+E_-(\omega,\tau_0,\xi)]}{\sqrt{\omega/2}[\gamma+i\gamma^{-1}][E_+(\omega,\tau_0,1)+E_-(\omega,\tau_0,1)]}$$

Now, considering that

$$\frac{(1+i\omega\tau_0)}{\sqrt{\omega/2}[\gamma+i\gamma^{-1}]}=\sqrt{2\tau_0}\frac{(\gamma+z\gamma^{-1})+i(z\gamma-\gamma^{-1})}{\sqrt{z}[\gamma^2-\gamma^{-2}]}$$

from Eq. (B.1)

$$\lim_{z\to\infty}\big[\gamma^2-\gamma^{-2}\big]=-2z$$
$$\lim_{z\to\infty}\big(\gamma+z\gamma^{-1}\big)=\sqrt{2z}z$$
$$\lim_{z\to\infty}\big(z\gamma-\gamma^{-1}\big)=\frac{z}{\sqrt{2z}}-\sqrt{2z}=-\frac{\sqrt{z}}{\sqrt{2}}$$

and

$$\lim_{z\to\infty}\frac{(1+i\omega\tau_0)}{\sqrt{\omega/2}[\gamma+i\gamma^{-1}]}=\sqrt{2\tau_0}\frac{\sqrt{2z}z-i(\sqrt{z}/\sqrt{2})}{-2z\sqrt{z}}$$
$$=\sqrt{2\tau_0}\left(\frac{-1}{\sqrt{2}}+i0\right)=-\sqrt{\tau_0}$$

then the following asymptotic formulas hold:

$$\lim_{\omega\to\infty}I_0(\omega,\tau_0,\xi)=-\sqrt{\tau_0}\frac{L}{k}\lim_{z\to\infty}\frac{[E_+(\omega,\tau_0,1-\xi)+E_-(\omega,\tau_0,1-\xi)]}{[E_+(\omega,\tau_0,1)+E_-(\omega,\tau_0,1)]}$$
$$\lim_{\omega\to\infty}I_1(\omega,\tau_0,\xi)=-\sqrt{\tau_0}\frac{L}{k}\lim_{z\to\infty}\frac{[E_+(\omega,\tau_0,\xi)+E_-(\omega,\tau_0,\xi)]}{[E_+(\omega,\tau_0,1)+E_-(\omega,\tau_0,1)]}$$

The asymptotic value of the functions

$$\Psi_0=\frac{[E_+(\omega,\tau_0,1-\xi)+E_-(\omega,\tau_0,1-\xi)]}{[E_+(\omega,\tau_0,1)+E_-(\omega,\tau_0,1)]}$$
$$\Psi_1=\frac{[E_+(\omega,\tau_0,\xi)+E_-(\omega,\tau_0,\xi)]}{[E_+(\omega,\tau_0,1)+E_-(\omega,\tau_0,1)]}$$

can be evaluated by observing that

$$\lim_{\omega\to\infty}\Lambda=\lim_{\omega\to\infty}\sqrt{\frac{\omega}{2}}\big[\gamma+i\gamma^{-1}\big]$$
$$=\lim_{\omega\to\infty}\frac{1}{\sqrt{2\tau_0}}\sqrt{z}\left[\frac{1}{\sqrt{2z}}+i\sqrt{2z}\right]$$
$$=\frac{1}{\sqrt{2\tau_0}}\left(\frac{1}{\sqrt{2}}+i\sqrt{2}z\right)$$

then

$$\lim_{\omega\to\infty}\big[E_+(\omega,\tau_0,\xi)+E_-(\omega,\tau_0,\xi)\big]$$
$$=\lim_{\omega\to\infty}\big[e^{\Lambda\xi}+e^{-\Lambda\xi}\big]$$
$$=\lim_{\omega\to\infty}\big\{e^{\Lambda_r\xi}\big[\cos(\Lambda_i\xi)+i\sin(\Lambda_i\xi)\big]+e^{-\Lambda_r\xi}\big[\cos(\Lambda_i\xi)-i\sin(\Lambda_i\xi)\big]\big\}$$
$$=\lim_{\omega\to\infty}\big\{2\mathrm{Ch}(\Lambda_r\xi)\cos(\Lambda_i\xi)+i2\mathrm{Sh}(\Lambda_r\xi)\sin(\Lambda_i\xi)\big\}$$
$$=\lim_{\omega\to\infty}\left\{2\mathrm{Ch}\left(\frac{1}{2\sqrt{\tau_0}}\xi\right)\cos\left(\frac{z}{\sqrt{\tau_0}}\xi\right)+i2\mathrm{Sh}\left(\frac{1}{2\sqrt{\tau_0}}\xi\right)\sin\left(\frac{z}{\sqrt{\tau_0}}\xi\right)\right\}$$

and

$$\lim_{\omega\to\infty} \Psi_0(\xi,\omega) = \frac{\{\mathrm{Ch}(\frac{1}{2\sqrt{\tau_0}}(1-\xi))\cos(\frac{z}{\sqrt{\tau_0}}(1-\xi))+i\,\mathrm{Sh}(\frac{1}{2\sqrt{\tau_0}}(1-\xi))\sin(\frac{z}{\sqrt{\tau_0}}(1-\xi))\}}{\{\mathrm{Ch}(\frac{1}{2\sqrt{\tau_0}})\cos(\frac{z}{\sqrt{\tau_0}})+i\,\mathrm{Sh}(\frac{1}{2\sqrt{\tau_0}})\sin(\frac{z}{\sqrt{\tau_0}})\}}$$

$$\lim_{\omega\to\infty} \Psi_1(\xi,\omega) = \frac{\{\mathrm{Ch}(\frac{1}{2\sqrt{\tau_0}}\xi)\cos(\frac{z}{\sqrt{\tau_0}}\xi)+i\,\mathrm{Sh}(\frac{1}{2\sqrt{\tau_0}}\xi)\sin(\frac{z}{\sqrt{\tau_0}}\xi)\}}{\{\mathrm{Ch}(\frac{1}{2\sqrt{\tau_0}})\cos(\frac{z}{\sqrt{\tau_0}})+i\,\mathrm{Sh}(\frac{1}{2\sqrt{\tau_0}})\sin(\frac{z}{\sqrt{\tau_0}})\}}$$

It is then easy to see that

$$\lim_{\omega\to\infty} \Psi_0(0,\omega)=1, \qquad \lim_{\omega\to\infty} \Psi_1(1,\omega)=1$$

whereas the asymptotic behaviour for the other values of ξ is oscillatory. For example,

$$\begin{aligned}\lim_{\omega\to\infty} \Psi_0(1,\omega) &= \frac{1}{\{\mathrm{Ch}(\frac{1}{2\sqrt{\tau_0}})\cos(\frac{z}{\sqrt{\tau_0}})+i\,\mathrm{Sh}(\frac{1}{2\sqrt{\tau_0}})\sin(\frac{z}{\sqrt{\tau_0}})\}} \\ &= \frac{\{\mathrm{Ch}(\frac{1}{2\sqrt{\tau_0}})\cos(\frac{z}{\sqrt{\tau_0}})-i\,\mathrm{Sh}(\frac{1}{2\sqrt{\tau_0}})\sin(\frac{z}{\sqrt{\tau_0}})\}}{\{\mathrm{Ch}^2(\frac{1}{2\sqrt{\tau_0}})\cos^2(\frac{z}{\sqrt{\tau_0}})+\mathrm{Sh}^2(\frac{1}{2\sqrt{\tau_0}})\sin^2(\frac{z}{\sqrt{\tau_0}})\}} \\ &= \frac{\{\mathrm{Ch}(\frac{1}{2\sqrt{\tau_0}})\cos(\frac{z}{\sqrt{\tau_0}})-i\,\mathrm{Sh}(\frac{1}{2\sqrt{\tau_0}})\sin(\frac{z}{\sqrt{\tau_0}})\}}{\mathrm{Ch}^2(\frac{1}{2\sqrt{\tau_0}})-\sin^2(\frac{z}{\sqrt{\tau_0}})}\end{aligned}$$

Appendix C

The function

$$\begin{aligned}\Xi(\omega) &= -\mathrm{Re}\{\Gamma_0(\omega)\}/\mathrm{Im}\{\Gamma_0(\omega)\} \\ &= -\mathrm{Re}\{I_0(\omega,0)\}/\mathrm{Im}\{I_0(\omega,0)\}\end{aligned}$$

assumes a general behaviour for $Bi=0$ when expressed as a function of $z=\omega\tau_0$, in fact,

$$\begin{aligned}I_0(\omega,0) &= R_+(\omega)[E_+(\omega,\tau_0,1)+E_-(\omega,\tau_0,1)] \\ &= \frac{L}{k}\frac{[E_+(\omega,\tau_0,1)+E_-(\omega,\tau_0,1)]}{\widetilde{\Lambda}(\omega,\tau_0,0)[E_+(\omega,\tau_0,1)+E_-(\omega,\tau_0,1)]} \\ &= \frac{L}{k\widetilde{\Lambda}(\omega,\tau_0,0)}\end{aligned}$$

and for the case $\beta=0$, $Bi=0$,

$$\begin{aligned}\widetilde{\Lambda}(\omega,\tau_0,0) &= \frac{1}{\sqrt{\tau_0}(1+i\omega\tau_0)}\sqrt{\frac{\omega\tau_0}{2}}[\gamma(\omega\tau_0)+i\gamma^{-1}(\omega\tau_0)] \\ &= \frac{\sqrt{z}}{\sqrt{2\tau_0}}\frac{[\gamma(z)+i\gamma^{-1}(z)]}{(1+iz)}\end{aligned}$$

and

$$\begin{aligned}I_0(\omega,0) &= \frac{L}{k\widetilde{\Lambda}(\omega,\tau_0,0)} = \frac{L}{k}\frac{\sqrt{2\tau_0}(1+iz)}{\sqrt{z}[\gamma(z)+i\gamma^{-1}(z)]} \\ &= \frac{L}{k}\frac{\sqrt{2\tau_0}(1+iz)[\gamma(z)-i\gamma^{-1}(z)]}{\sqrt{z}[\gamma^2(z)+\gamma^{-2}(z)]} \\ &= \sqrt{2\tau_0}\frac{L}{k}\frac{[\gamma(z)+z\gamma^{-1}(z)]+i[z\gamma(z)-\gamma^{-1}(z)]}{\sqrt{z}[\gamma^2(z)+\gamma^{-2}(z)]}\end{aligned}$$

and then

$$\begin{aligned}\Xi(\omega) &= -\frac{\mathrm{Re}\{I_0(\omega,0)\}}{\mathrm{Im}\{I_0(\omega,0)\}} = -\frac{[\gamma(z)+z\gamma^{-1}(z)]}{[z\gamma(z)-\gamma^{-1}(z)]} \\ &= -\frac{[\gamma^2(z)+z]}{[z\gamma^2(z)-1]} = -\frac{[\sqrt{1+z^2}-z+z]}{[z\sqrt{1+z^2}-z^2-1]} \\ &= \frac{1}{[\sqrt{1+z^2}-z]} = \gamma^{-2}(z)\end{aligned}$$

and (see Appendix B) the asymptotic behaviour of $\Xi(\omega)$ is

$$\lim_{z\to\infty} \Xi(z) = 2z.$$

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